Mathematics 222B Lecture 12 Notes

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1 Overview of Schauder Theory

1.1 Main theorems of Schauder theory

Schauder theory can be summarized as "Hölder-based elliptic regularity theory." Here are some of the main theorems.

Theorem 1.1 (Shauder, interior regularity, divergence form)**.** *Let U be an open subset of $\mathbb{R}^d$, and suppose that $Pu = f$, where $Pu = -\partial_j(a^{j,k}\partial_k u)$, $a \succ \lambda I$, and $a \in C^{k-1,\alpha}(\overline{U})$, Assume that $u \in C^{k,\alpha}(\overline{U})$ (with $k \geq 1$ and $0 < \alpha < 1$) and $f \in C^{k-2,\alpha}(\overline{U})$ (if $k = 1$, we assume that $f = f^0 + \sum_{j=1}^d \partial_j f^j$ with $f^0, f^j \in C^{0,\alpha}(\overline{U})$). Then for all $V \subseteq\subseteq U$, there exists a constant $C = C_V$ such that*

$$\|u\|_{C^{k,\alpha}(V)} \leq C(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}).$$

(If $k = 1$, we define $\|f\|_{C^{-1,\alpha}} := \|f^0\|_{C^{0,\alpha}} + \sum_{j=1}^d \|f^j\|_{C^{0,\alpha}}$.)

Remark 1.1. We omit the $b^j + \partial_j u + cu$ parts because they can be easily added, and they are generally dealt with on a case-by-case basis to determine what regularity you need for b and c.

Theorem 1.2 (Schauder, interior regularity, non-divergence form)**.** *Let U be an open subset of $\mathbb{R}^d$, and suppose that $Qu = f$, where $Qu = -a^{j,k}\partial_j\partial_k u$, $a \succ \lambda I$, and $a \in C^{k-2,\alpha}(\overline{U})$, Assume that $u \in C^{k,\alpha}(\overline{U})$ (with $k \geq 2$ and $0 < \alpha < 1$) and $f \in C^{k-2,\alpha}(\overline{U})$. Then for all $V \subseteq\subseteq U$, there exists a constant $C = C_V$ such that*

$$\|u\|_{C^{k,\alpha}(V)} \leq C(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}).$$

Definition 1.1. We say that U has $C^{k,\alpha}$ **boundary** if for all $x \in \partial U$, there exists an $r > 0$ such that (after possibly rearranging the axes)

$$U \cap B_r(x) = \{y \in B_r(x) : y^n > \gamma(y^1, \dots, y^{d-1})\gamma \in C^{k,\alpha}\}.$$

Theorem 1.3 (Schauder, boundary regularity, divergence form)**.** *Assume the same hypotheses in the interior divergence form theorem, and assume that ∂U is $C^{k,\alpha}$ and U is bounded. Take $Pu = f$ with the boundary condition $u|_{\partial U} = 0$. Then there exists a constant C such that*

$$\|u\|_{C^{k,\alpha}(U)} \leq C(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}).$$

Theorem 1.4 (Schauder, boundary regularity, non-divergence form)**.** *Assume the same hypotheses in the interior non-divergence form theorem, and assume that ∂U is $C^{k,\alpha}$ and U is bounded. Take $Qu = f$ with the boundary condition $u|_{\partial U} = 0$. Then there exists a constant C such that*

$$\|u\|_{C^{k,\alpha}(U)} \leq C(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}).$$

1.2 Overall strategies of the proofs

Here are strategies to prove these theorems.

Interior:

1. Prove the result in the constant coefficient case ($a^{j,k}$ constant).
2. Prove the general case using the constant coefficient case by the **method of freezing the coefficients**: Elliptic regularity is local, so we can split the space into small balls and prove the statement on each ball. The regularity of $a^{j,k}$ allows us to approximate the general problem by constant coefficient problems.

Boundary:

0. Locally straighten the boundary to reduce to the case of half balls.

1+2. Use the same method as for interior regularity. Step 0 makes the relevant constant coefficient problems be the half-space case.

We will provide two proofs for the constant coefficient case:

A. Littlewood-Paley theory proof

B. Compactness + contradiction proof.

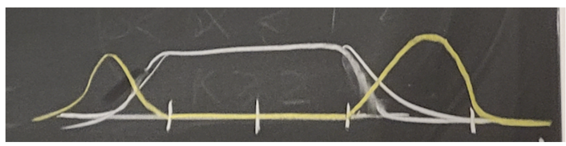
1.3 Littlewood-Paley proof of Schauder estimates

Theorem 1.5 (Constant coefficient Schauder estimate)**.** *Let $Pu = -\partial_j(a_0^{j,k}\partial_k u) = -a_0^{j,k}\partial_j\partial_k u$, where $a_0^{j,k}$ is constant on $\mathbb{R}^d$, and $a_0 \succ \lambda I$. Assume that $|a_0^{j,k}| \leq \Lambda$, where $\Lambda \geq \lambda > 0$. For $u \in C_c^{k,\alpha}(\mathbb{R}^d)$ and $f \in C^{k-2,\alpha}(\mathbb{R}^d)$ such that $Pu = f$,*

$$\|u\|_{C^{k,\alpha}(\mathbb{R}^d)} \leq C\|f\|_{C^{k-2,\alpha}(\mathbb{R}^d)}.$$

Let us emphasize that we assume that u has *compact support*. We will focus on the case $k = 2$.

Definition 1.2. Define
$$\chi_{\le 0}(\xi) = \begin{cases} 1 & |\xi| \le 1 \\ 0 & |\xi| > 1 \\ \ge 0 & \forall \xi, \end{cases}$$
$$\chi_{\le k}(\xi) = \chi_{\le 0}(\xi/2^k),$$
$$\chi_k(\xi) = \chi_{\le k+1}(\xi) - \chi_{\le k}(\xi) \qquad (\text{so } \operatorname{supp} \chi_k \subseteq \{\xi : 2^k \le |\xi| \le 2^{k+2}\}).$$
The **Littlewood-Paley projections** are
$$P_k v = \mathcal{F}^{-1}(\chi_k(\xi)\widehat{v}), \qquad P_{\le k} = \mathcal{F}^{-1}(\chi_{\le k}(\xi)\widehat{v}).$$



Observe that for all $v \in \mathcal{S}'(\mathbb{R}^d)$,
$$v = P_{\le k_0} v + \sum_{k > k_0} P_k v.$$
If v satisfies certain regularirt conditions in the same norm, $P_{\le k_0} v \to 0$ as $k_0 \to -\infty$. Note that $|\xi| \simeq 2^k$ on $\operatorname{supp} \chi_k$.

Lemma 1.1 (Littlewood-Paley characterization of $C^{0,\alpha}(\mathbb{R}^d)$)**.** *Let $v \in C^{0,\alpha}(\mathbb{R}^d)$. Then*
$$[v]_{C^{0,\alpha}} = \sup_{\substack{x,y \\ x \ne y}} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \simeq \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|P_k v\|_{L^\infty}.$$

Here is the proof of this lemma:

Proof. ($\gtrsim$): Both seminorms are invariant to scaling, so it suffices to consider $k = 0$. So we just have to show that
$$|P_0 v| \lesssim [v]_{C^{0,\alpha}}.$$
Since $\int \overset{\vee}{\chi}_0(y)\, dy = 0$ iff $\chi_0(0) = 0$,
$$P_0 v = \int \overset{\vee}{\chi}_0(x-y) v(y)\, dy = \int \overset{\vee}{\chi}_0(x-y)(v(y) - v(x))\, dy$$

$$\leq \underbrace{\int \overset{\vee}{\chi}_0(x-y)|x-y|^\alpha \, dy}_{\text{fixed } \mathcal{S}(\mathbb{R}^d) \text{ function}}[v]_{C^{0,\alpha}}.$$

($\lesssim$): Whenever we work with Littlewood-Paley theory, we should think about what scale we are working on. Let $L = |x-y|$, and choose k_0 so that $L^{-1} \simeq 2^{k_0}$. Decompose

$$v(x) - v(y) = P_{\leq k_0} v(k) - P_{\leq k_0} v(y) + \sum_{k > k_0} P_k v(x) - P_k v(y)$$

We can bound the latter two terms as

$$\begin{aligned}
\left\| \sum_{k \geq k_0} P_{\leq k_0} v \right\|_{L^\infty} &\leq \sum_{k < k_0} \|P_k v\|_{L^\infty} \\
&\leq \sum_{k \geq k_0} 2^{-k\alpha} [v]_{C^{0,\alpha}} \\
&\simeq L^\alpha [v]_{C^{0,\alpha}}.
\end{aligned}$$

We can bound the first terms using the fundamental theorem of calculus:

$$\begin{aligned}
|P_{\leq k_0} v(x) - P_{\leq k_0} v(y)| &\leq |\nabla P_{\leq k_0} v\|_{L^\infty} L \\
&\leq \sum_{k \leq k_0} \|\nabla P_k v\|_{L^\infty} L \\
&\lesssim L \sum_{k \leq k_0} 2^k 2^{-k\alpha} [v]_{wtC^{0,\alpha}} \\
&\simeq L L^{-(1-\alpha)} [v]_{\widetilde{C}^{0,\alpha}}.
\end{aligned}$$

□

Now we can prove the theorem.

Proof. We have $P(P_k u) - P_k f$, so

$$a^{j,\ell} \xi_j \xi_\ell \widehat{P_k u} = \widehat{P_k f}.$$

Since $\lambda |\xi|^2 \leq a_0^{j,\ell} \xi_j \xi_k$,

$$\widehat{P_k u} = \frac{2^{2k}}{a^{j,\ell} \xi_j \xi_\ell} \widehat{P_k f} \widetilde{\chi}_k \frac{1}{2^{2k}} = \frac{1}{2^{2k}} \underbrace{\frac{2^{2k}}{a^{j,\ell} \xi_j \xi_\ell} \widehat{\chi}_k}_{\eta_k(\xi)} \widehat{P_k f},$$

where $\widetilde{\chi}_k = 1$ on $\operatorname{supp} \chi_k$ and $\operatorname{supp} \widetilde{\chi}_k \subseteq \{|\xi| \simeq 2^k\}$. Then

$$P_k u = 2^{-2k} \overset{\vee}{\eta}_k * P_k f,$$

so

$$\|P_k u\|_{L^\infty} \leq C 2^{-2k} \|P_k f\|_{L^\infty} \leq C 2^{-2k - \chi k} [f],$$

which completes the proof. □

1.4 Compactness and contradiction proof of Schauder estimates

Proof. Here are the steps:

1. Assume that the desired inequality fails. Then there exist $a_n^{j,k}, u_n, f_n$ such that (after normalization)

$$P_n u_n = f_n, \qquad [u_n]_{C^{2\alpha}} = 1, \qquad [f_n]_{C^{0,\alpha}} \leq \frac{1}{n}.$$

   After translation, we may also ensure that for some $\eta_n \in \mathbb{R}^d$,

$$|D^2 u_n(\eta_n) - D^2 u_n(0)| \geq c|\eta_n|^\alpha.$$

   Using scaling, we can assume that $|\eta_n| = 1$.

2. Another massaging: Define $v_n(x) = u_n(x) - u_n(0) - xDu_n(0) - \frac{1}{2}x^2 D^2 u_n(0)$ to make $D^2 v_n(0) = 0$. Then

$$P_n v_n = \wr f_n, \qquad \widetilde{f}_n \to 0, [D^2 v_n]_{C^{0,\alpha}} = 1, \qquad |D^2 v_n(\eta_j)| \geq c.$$

3. Take the limit: Let $a_n^{j,k} \to a_\infty^{j,k}$, $\widetilde{f}_n \to 0$, $v_n \to v$, and $\eta_n \to \eta_\infty$. Then $P_\infty v = 0$ on $\mathbb{R}^d$, while

$$[D^2 v]_{C^{0,\alpha}} \leq 1, \qquad D^2 v(\eta_\infty) \neq 0.$$

   But now use Liouville's theorem for P_∞ (using Liouville's theorem for the Laplace equation) to get that $D^2 v(\eta_\infty) = 0$, a contradiction. □